The Chinese University of Hong Kong
Department of Mathematics
2024 Enrichment Programme for Young Mathematics Talents

TOWARDS DIFFERENTIAL GEOMETRY
Test 2, 23/08/2024

Instructions:

- Time allowed: $\mathbf{90 \pm \delta}$ **minutes.**
- This paper consists of **Basic Part**, **Harder Part** and **Bonus Part**.
- The full mark of the paper is **80 points** and bonus mark **15 points**.
- Answer **ALL** questions in Basic Part and **THREE** questions in Harder Part. Make your best effort to answer the Bonus Part.
- Show your work clearly and concisely. Give thorough explanation and justification for your calculations and observations.
- Write your answers in the spaces provided in the Answer Booklet. Begin each question on a new page. Clearly indicate the question number in the designated slot at the top of each page.
- Supplementary answer sheets and rough paper will be supplied on request.
- Non-graphical calculators are allowed.
- Unless otherwise specified, numerical answers must be exact.

Full Name: ______________________________ Group: ____________________

Basic Part (50 points). Answer **ALL** questions in this part.

1. (10 points) Compute the curvature $\kappa(t)$ of the following curves:

   (a) (4 points) $\alpha(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$, for $t \in (0,1)$.

   (b) (3 points) $\alpha(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right)$, for $t \in (0, 2\pi)$.

   (c) (3 points) $\alpha(t) = (t, \cosh t)$, for $t \in (0, \infty)$.

2. (20 points) Given the parameterized curve

$$\mathbf{r}(t) = \left(e^t \cos t, e^t \sin t, 2e^t\right), \quad t \in (0, \infty).$$

   (a) (5 points) Find $\mathbf{r}'(t)$, hence find $\| \mathbf{r}'(t) \|$.

   (b) (2 points) Find $s := \int_0^t \| \mathbf{r}'(u) \| du$.

   (c) (3 points) Hence, or otherwise, find an arc-length parameterization of $\mathbf{r}(t)$. Please state clearly the domain of s.

   (d) (10 points) Hence, or otherwise find the **TNB** frame, the curvature κ, and torsion τ. (You may find your answer with the original parameterization, or with the arc-length parameterization you found.)

3. (20 points) It is given that the parametrization of the torus is

$$\mathbf{X}(\phi, \theta) = ((R + r\sin\phi)\cos\theta, (R + r\sin\phi)\sin\theta, r\cos\phi), 0 < \phi < 2\pi, 0 < \theta < 2\pi$$

   where $R > r > 0$ and R, r are constants.

   (a) (6 points) Compute $\mathbf{X}_\phi \times \mathbf{X}_\theta$.

   (b) (4 points) Hence, prove that the torus is a regular surface.

   (c) (5 points) Compute the first fundamental form of $\mathbf{X}$ as a 2×2 matrix.

   (d) (5 points) Show that the surface area of the torus is $4\pi^2 rR$.

Harder Part – Curve Theory (15 points).

Answer **ONE** question in **Curve Theory** section

Curve Theory – Structured Questions

4. (15 points) This question investigates vanishing curvature κ and torsion τ

   (a) (6 points) Let $\mathbf{r}(t)$ be a regular parametrised curve. Prove that if its curvature satisfies $\kappa(t) = 0$ of any $a < t < b$, then $\mathbf{r}(t)$ is a straight line.

   (b) (9 points) Let $\mathbf{r}(s)$ be a regular parametrised curve with $\| \mathbf{r}'(s)\| = 1$. Further suppose that there exists a constant $\mathbf{v} \in \mathbb{R}^3, C \in \mathbb{R}$ such that $\langle \mathbf{r}(s), \mathbf{v}\rangle = C$ for all s.

   (i) (4 points) Differentiating with respect to s, prove that

   $$\langle \mathbf{r}'(s), \mathbf{v}\rangle = \langle \mathbf{r}''(s), \mathbf{v}\rangle = \langle \mathbf{r}'''(s), \mathbf{v}\rangle = 0.$$

   (ii) (3 points) Explain why $\langle \mathbf{r}'(s) \times \mathbf{r}''(s), \mathbf{r}'''(s)\rangle = 0$.

   (iii) (2 points) Compute torsion $\tau(s)$ of $\mathbf{r}(s)$.

5. (15 points) This question is investigates curves on sphere

   Let $\alpha(s)$ be a regular space curve with arc length parameterization. $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are the unit tangent, unit normal and unit binormal to the curve respectively. Let $\kappa(s)$ and $\tau(s)$ be the curvature and torsion of the curve. Suppose $\alpha(s)$ lies on the some sphere centred at the origin for any s, that is:

   $$\|\alpha(s)\| = r \text{ for some } r \in \mathbb{R}^+$$

   (a) (4 points) Differentiating with respect to s, show that

   $$\langle \alpha, \alpha'\rangle = 0 \text{ and } \langle \alpha', \alpha'\rangle = -\langle \alpha'', \alpha\rangle \text{ and } \langle \alpha''', \alpha\rangle = 0.$$

   (b) (3 points) Using (a) (ii) and Frenet Serret Equation, show that $\langle \alpha(s), \mathbf{N}(s)\rangle = -\dfrac{1}{\kappa(s)}$.

   (c) (3 points) Using Frenet Serret Equation, show that

   $$\alpha'''(s) = \frac{d}{ds}(\kappa(s)\,\mathbf{N}(s)) = -\kappa^2(s)\,\mathbf{T}(s) + \kappa'(s)\,\mathbf{N}(s) + \kappa(s)\tau(s)\,\mathbf{B}(s).$$

   (d) (i) (3 points) By considering $\langle \alpha'''(s), \alpha(s)\rangle = 0$, compute $\langle \alpha(s), \mathbf{B}(s)\rangle$

   (ii) (2 points) Deduce that $\alpha(s) = -\dfrac{1}{\kappa(s)}\mathbf{N}(s) + \dfrac{\kappa'(s)}{\kappa^2(s)\tau(s)}\mathbf{B}(s)$.
   (**Hints:** $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ constitutes an orthonormal basis.)

Harder Part – Surface Theory (5+10 points).

Answer **ALL** True/False Questions. Answer **ONE** question in **Surface Theory** section.

General notations:

- Let $\mathbf{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a regular parametrized surface. We name the surface as M.
- Unit normal vector $\mathbf{n}$ of $\mathbf{X}$ is computed by $\mathbf{n} = \dfrac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}$.

Surface Theory – True/False Questions

6. (5 points) Mark each of the following statements "True" (meaning that it is a true statement) or "False" (meaning that there are counterexamples to the statement). No reasoning is required and each question carries 1 point.

   (a) A curve C lying on a regular surface M must be a regular curve.

   (b) A connected subset of a regular surface must be a regular surface.

   (c) The first fundamental form of a regular surface as a 2×2 matrix must be invertible.

   (d) Let M be a surface obtained by revolving the curve $z = f(x)$ around the z-axis. Then $(x\cos\theta, x\sin\theta, f(x))$, $0 < \theta < 2\pi$ gives a parametrization of M.

   (e) Surface area of M is given by: $\iint_D \sqrt{\det(I)}dA$, where I is the first fundamental form of $\mathbf{X}$ and dA is the area element of M.

Surface Theory – Structured Questions

7. (10 points) Let $\mathbf{a} \in \mathbb{R}^3$ be a fixed vector such that for every $(u, v) \in D$

$$\| \mathbf{X}(u, v) - \mathbf{a} \| = r \qquad \ldots (\#)$$

   (a) (1 point) Explain the geometric meaning of (#).

   (b) (3 points) Prove that $\langle \mathbf{X}_u, \mathbf{X} - \mathbf{a} \rangle = \langle \mathbf{X}_v, \mathbf{X} - \mathbf{a} \rangle = 0$.

   (c) (3 points) Prove that $\mathbf{X}(u, v) - \mathbf{a}$ is parallel to $\mathbf{X}_u \times \mathbf{X}_v$.

   (d) (3 points) Deduce that all normal vectors of $\mathbf{X}$ passes through $\mathbf{a}$.

8. (10 points) Let $\mathbf{a} \in \mathbb{R}^3$ be a fixed vector such that for every $(u, v) \in D$

$$\mathbf{X}(u, v) - \mathbf{a} = \lambda(u, v)\,\mathbf{n} \qquad \ldots (\#\#)$$

   for some smooth functions $\lambda : U \subset \mathbb{R}^2 \to \mathbb{R}$

   (a) (1 point) Explain the geometric meaning of (##).

   (b) (2 points) Show that $\| \mathbf{X} - \mathbf{a} \|^2 = \lambda^2$.

   (c) (4 points) Using the fact that $\langle \mathbf{X}_u, \mathbf{n} \rangle = \langle \mathbf{X}_v, \mathbf{n} \rangle = 0$, prove that

$$\frac{\partial}{\partial u}\lambda^2 = \frac{\partial}{\partial v}\lambda^2 = 0.$$

   (d) (3 points) Deduce that $\mathbf{X}$ is part of a sphere. What is the radius of the sphere?

Bonus Part (15 points). Try your best to answer the question in this part.

9. This question will investigate the tangent plane of a surface and the geometric application of the first fundamental form.

   Let S be a regular surface parameterized by $\mathbf{X}(u, v)$ for any $(u, v) \in U$. For any point $p := \mathbf{X}(u_0, v_0)$, we can define a **tangent space** T_pS of S at p by

   $$T_pS := \{\alpha\,\mathbf{X}_u(u_0, v_0) + \beta\,\mathbf{X}_v(u_0, v_0) : \alpha, \beta \in \mathbb{R}\}.$$

   (a) (5 points) Show that for any $(u_0, v_0) \in U \subseteq \mathbb{R}^2$, $\mathbf{x} = \alpha\,\mathbf{X}_u(u_0, v_0) + \beta\,\mathbf{X}_v(u_0, v_0) \in T_pS$, we have

   $$\langle \mathbf{x}, \mathbf{x} \rangle = \begin{pmatrix} \alpha & \beta \end{pmatrix} I_p \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

   where I_p denotes the first fundamental form as a 2×2 matrix evaluated at p.

   (b) (5 points) Let $\mathbf{r} : V \subseteq \mathbb{R} \to S$, be a regular parameterized curve on the surface S defined by

   $$\mathbf{r}(t) := \mathbf{X}(f(t), g(t)) \text{ for any } t \in V$$

   and let $s = s(t)$ be the arc-length of $\mathbf{r}$. Given that

   $$\mathbf{r}'(t) = f'(t)\,\mathbf{X}_u(f(t), g(t)) + g'(t)\,\mathbf{X}_v(f(t), g(t)),$$

   show that

   $$\left(\frac{ds}{dt}\right)^2 = \begin{pmatrix} f'(t) & g'(t) \end{pmatrix} I \begin{pmatrix} f'(t) \\ g'(t) \end{pmatrix}.$$

   (c) (5 points) Now let $\mathbf{Y}(u, v)$ be a regular parameterized surface with first fundamental form

   $$I = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(v) \end{pmatrix}.$$

   Let $\alpha(t) := \mathbf{Y}(\sin(\log t), \log t)$ for $t \in (1, e^\pi)$. Find the arc-length of $\alpha(t)$.

~~ END OF PAPER ~~

Formula Sheet

1 Notations

1. $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ be a space curve (not necessarily parameterised by arc-length)
   - $\mathbf{r}(t)$ denotes a space curve **not necessarily** parametrized by arc-length
   - $\mathbf{r}(s)$ denotes a space curve parametrized by arc-length
2. $\mathbf{T}$ denotes the unit tangent of $\mathbf{r}$
3. $\mathbf{N}$ denotes the unit normal of $\mathbf{r}$
4. $\mathbf{B}$ denotes the unit binormal of $\mathbf{r}$
5. κ denotes the curvature of $\mathbf{r}$
6. τ denotes the torsion of $\mathbf{r}$

2 Formulae of T, N, B Frame

(1). $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ and $\mathbf{T}(s) = \mathbf{r}'(s)$ $\quad [\mathbf{T}'(s) = \mathbf{r}''(s)]$

(2). $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ and $\mathbf{N}(s) = \dfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \dfrac{\mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}$

(3). $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ and $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$

3 Formulae of Curvature κ

(1). $\kappa(t) = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$

(2). $\kappa(s) = \|\mathbf{T}'(s)\| = \|\mathbf{r}''(s)\|$

(3). $\kappa(t) = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$

(4). $\kappa(t) = \dfrac{\left|x'(t)y''(t) - x''(t)y'(t)\right|}{\left|x'(t)^2 + y'(t)^2\right|^{3/2}}$, for plane curve $\mathbf{r}(t) = \big(x(t), y(t)\big)$

(5). $\kappa(x) = \dfrac{|f''(x)|}{\big(1 + (f'(x))^2\big)^{3/2}}$, for graph of function $y = f(x)$

(6). $\kappa(\theta) = \dfrac{|r(\theta)^2 + 2r'(\theta)^2 - r(\theta)r''(\theta)|}{\big(r(\theta)^2 + r'(\theta)^2\big)^{3/2}}$, for $\mathbb{R}^2$ polar graph $\mathbf{r}(\theta) = \big(r(\theta)\cos\theta, r(\theta)\sin\theta\big)$

4 Formulae for Torsion τ

(1). $\tau(t) = \left\langle \frac{\mathbf{N}'(t)}{\|\mathbf{r}'(t)\|}, \mathbf{B}(t) \right\rangle$ and $\tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s) \rangle = -\langle \mathbf{N}(s), \mathbf{B}'(s) \rangle$

(2). $\tau(t) = \frac{\langle \mathbf{r}'(t) \times \mathbf{r}''(t), \mathbf{r}'''(t) \rangle}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}$

5 Frenet Serret Equation

(1). $$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}$$

(2). $$\begin{pmatrix} \mathbf{T}'(t) \\ \mathbf{N}'(t) \\ \mathbf{B}'(t) \end{pmatrix} = \begin{pmatrix} 0 & \|\mathbf{r}'(t)\|\kappa(t) & 0 \\ -\|\mathbf{r}'(t)\|\kappa(t) & 0 & \|\mathbf{r}'(t)\|\tau(t) \\ 0 & -\|\mathbf{r}'(t)\|\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(t) \\ \mathbf{N}(t) \\ \mathbf{B}(t) \end{pmatrix}$$